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LETTER TO THE EDITOR

Geometrical optics in nonlinear media and integrable equations

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Abstract
It is shown that the geometrical optics limit of the Maxwell equations for certain nonlinear media with slow variation along one axis and particular dependence of the dielectric constant on frequency and fields gives rise to the dispersionless Veselov–Novikov equation for the refractive index. It is demonstrated that the dispersionless Veselov–Novikov hierarchy is amenable to the quasiclassical $\bar{\partial}$-dressing method. A connection is noted between the geometrical optics phenomena under consideration and the quasiconformal mappings on the plane.

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A great variety of nonlinear phenomena from various fields of physics, applied physics, mathematics and applied mathematics can be modelled by nonlinear integrable equations [1–8]. A subclass of such equations, the so-called dispersionless integrable equations, has attracted particular interest during the last decade [9–17].

In this letter we will show that the propagation of electromagnetic waves of high frequency in certain nonlinear media is governed by the dispersionless Veselov–Novikov (dVN) equation. Namely, we will demonstrate that the Maxwell equations describing the propagation of waves in media characterized by a slow variation along the z-axis and particular dependence of the dielectric constant on frequency and fields in the limit of geometrical optics give rise to the dVN equations for the refractive index. These equations provide us with integrable deformations of the plane eikonal equation which preserve, in particular, the total 'plane' squared refractive index $\iint n^2 \, \mathrm{d}x \, \mathrm{d}y$. Under more special conditions one gets the dispersionless Kadomtsev–Petviashvili (dKP) equation. We will show that the dVN equations are treatable by the quasiclassical $\bar{\partial}$-dressing method developed recently in [18–20]. The quasiclassical $\bar{\partial}$-dressing method also reveals a connection between the geometrical optics phenomena under consideration and the theory of quasiconformal mappings on the plane.

We start with the Maxwell equations in the absence of sources (we take the velocity of light $c = 1$)

$$\begin{aligned} &\nabla \wedge \mathbf{H} - \dot{\mathbf{D}} = 0 \qquad \nabla \cdot \mathbf{D} = 0 \\ &\nabla \wedge \mathbf{E} + \dot{\mathbf{B}} = 0 \qquad \nabla \cdot \mathbf{B} = 0 \end{aligned} \tag{1}$$

and the material equations of a medium

$$\mathbf{D} = \epsilon \mathbf{E} \qquad \mathbf{B} = \mu \mathbf{H}. \tag{2}$$

Here and below $\nabla = (\partial_x, \partial_y, \partial_z)$, while $\cdot$ and $\wedge$ denote the scalar and vector products respectively.

We will study the propagation of electromagnetic waves of the fixed, high frequency ω, i.e. we will look for the solutions of the Maxwell equations of the form [21]

$$\begin{aligned} &\mathbf{E}(x, y, z, t) = \mathbf{E_0}(x, y, z)\, \mathrm{e}^{-\mathrm{i}\omega t + \mathrm{i}\omega S(x,y,z)} \\ &\mathbf{H}(x, y, z, t) = \mathbf{H_0}(x, y, z)\, \mathrm{e}^{-\mathrm{i}\omega t + \mathrm{i}\omega S(x,y,z)} \end{aligned} \tag{3}$$

where $\mathbf{E_0}$, $\mathbf{H_0}$ and the phase $S(x, y, z)$ are certain functions.

In addition, we assume that the nonlinear medium is characterized (1) by the independence of the magnetic permeability $\mu(x, y, z)$ from the frequency ω; (2) by the Cole–Cole dependence [22]

$$\epsilon = \epsilon_0 + \frac{\tilde{\epsilon}}{1 + (\mathrm{i}\omega\tau_0)^{2\nu}} \qquad 0 < \nu < \frac{1}{2} \tag{4}$$

of the dielectric constant ϵ on ω (where $\epsilon_0(x, y, z)$ and τ_0 are independent of ω and $\tilde{\epsilon}$ is a function of coordinates and fields); (3) by a slow variation of all quantities along the z-axis such that $\frac{\partial}{\partial z} = \omega^{-\nu}\frac{\partial}{\partial \xi}$, where ξ is a 'slow' variable defined by $z = \omega^{\nu}\xi$, i.e.

$$S = S(x, y, \xi) \qquad \mathbf{E_0} = \mathbf{E_0}(x, y, \omega^{\nu}\xi) \qquad \mathbf{H_0} = \mathbf{H_0}(x, y, \omega^{\nu}\xi). \tag{5}$$

Now, rewriting equations (1) and (2) as the second-order differential equations for the electric and magnetic fields, respectively, using (3) and (5) and taking into account that at $\omega \to \infty$

$$\epsilon = \epsilon_0 + \frac{\epsilon_1}{\omega^{2\nu}} \tag{6}$$

where $\epsilon_1 = (\mathrm{i}\tau_0)^{2\nu}\tilde{\epsilon}$, one obtains in the leading ω^2 order

$$S_x^2 + S_y^2 = \mu\epsilon_0 \tag{7}$$

while, in the next $\omega^{2-2\nu}$ order, one obtains

$$S_\xi = (\mu\epsilon_1)^{\frac{1}{2}}. \tag{8}$$

Equation (7) is the standard eikonal equation on the x, y plane with the refractive index $n(x, y, \xi) = (\mu\epsilon_0)^{\frac{1}{2}}$. The slow variation of S along the z-axis is defined by equation (8), where $\mu\epsilon_1$ depends both on coordinates x, y, ξ and fields. We also assume that the absorption effects are negligible, so $\mu\epsilon$ can be taken as real. We would like to note that the first two features of the medium mentioned above are quite generic and they are valid for a variety of dielectric media [22–24], while the particular type of slow variations along the z-axis given by (8), seems, did not attract attention before.

Since the function ϵ_1 is determined by the geometrical optics limit $\omega \to \infty$ and the time translation symmetry $t \to t + \text{const}$ of the Maxwell equations for the solutions of type (3) is equivalent to the symmetry under transformations $S \to S + \text{const}$, one concludes that the function ϵ_1 for nonlinear media may depend on the coordinates x, y, ξ and only on S_x and S_y. Thus, the geometrical optics limit of the Maxwell equations in nonlinear media under consideration is governed by the equations

$$S_x^2 + S_y^2 = n^2(x, y, \xi) \tag{9}$$

$$S_\xi = \varphi(x, y, \xi; S_x, S_y) \tag{10}$$

where φ is some real-valued function. Introducing the variables $z = x + \mathrm{i}y, \bar{z} = x - \mathrm{i}y$ (please, do not confuse it with the variable used in equation (3)), one rewrites these equations as follows:

$$S_z S_{\bar{z}} = u(z, \bar{z}, \xi) \tag{11}$$

$$S_\xi = \varphi(z, \bar{z}; \xi, S_z, S_{\bar{z}}) \tag{12}$$

where $u = 4n^2$.

The compatibility of equations (9) and (10) (or (11), (12)) imposes constraints on the possible forms of the function φ, namely

$$S_{\bar{z}} \frac{\partial \varphi}{\partial z} + S_z \frac{\partial \varphi}{\partial \bar{z}} + u_z \varphi' + u_{\bar{z}} \varphi'' = u_\xi \tag{13}$$

where

$$\varphi' = \frac{\partial \varphi}{\partial S_z}(z, \bar{z}; S_z, S_{\bar{z}}) \qquad \varphi'' = \frac{\partial \varphi}{\partial S_{\bar{z}}}(z, \bar{z}; S_z, S_{\bar{z}}). \tag{14}$$

Here we restrict ourselves to functions φ which are polynomials in S_z, $S_{\bar{z}}$ and compatible with the real-valuedness of S and u. For the simplest choice $\varphi = \alpha_0(z, \bar{z}, \xi)$, equation (13) obviously gives $\alpha_0 = \text{const}$, i.e. $u_\xi = 0$, $S = \alpha_0 \xi + \tilde{S}(z, \bar{z})$. For the linear function $\varphi = \alpha S_z + \bar{\alpha} S_{\bar{z}} + \beta + \bar{\beta}$, one obtains $\alpha = \alpha(z)$, $\beta = \text{const}$, and

$$u_\xi = (\alpha u)_z + (\bar{\alpha} u)_{\bar{z}}. \tag{15}$$

For the quadratic $\varphi = \alpha S_z^2 + \bar{\alpha} S_{\bar{z}}^2 + \beta S_z + \bar{\beta} S_{\bar{z}} + \gamma + \bar{\gamma}$, equations (13) and (11) imply $\alpha = 0$, $\beta = \beta(z)$ and $\gamma = \text{const}$, i.e. one obtains the previous linear case.

The cubic $\varphi = \alpha S_z^3 + \bar{\alpha} S_{\bar{z}}^3 + \beta S_z^2 + \bar{\beta} S_{\bar{z}}^2 + \gamma S_z + \bar{\gamma} S_{\bar{z}} + \delta + \bar{\delta}$ obeys equations (13) and (11) if

$$\alpha = \alpha(z) \qquad \beta = 0 \qquad \gamma_{\bar{z}} = -\alpha_z u - 3\alpha u_z \qquad \delta = \text{const} \tag{16}$$

and one obtains

$$u_\xi = (\gamma u)_z + (\bar{\gamma} u)_{\bar{z}}. \tag{17}$$

In the particular case $\alpha = 1$ and, consequently $\gamma_{\bar{z}} = -3u_z$, equation (17) is nothing but the dispersionless Veselov–Novikov equation introduced in [11, 20].

In a similar manner one can construct nonlinear equations which correspond to higher degree polynomials φ. These higher degree cases apparently become physically relevant for the phenomena with large values of S_x and S_y. Thus, if we formally admit all possible degrees of S_z and $S_{\bar{z}}$ on the right-hand side of equation (12), then one has an infinite family of nonlinear equations, which may govern the ξ-variations of the wave fronts and 'refractive index' u. Since equation (12) should respect the symmetry $S \to -S$ of the eikonal equation (11), one readily concludes that only polynomials φ of the form $\sum_{n=1}^N u_n S_z^{2n-1} + \sum_{n=1}^N \bar{u}_n S_{\bar{z}}^{2n-1}$, are admissible (the constant terms which have appeared in the cases (15), (17) discussed above are, in fact, irrelevant). In the case $u_N = 1$ one obtains the dVN equation mentioned above ($N = 2$) and the so-called dVN hierarchy of nonlinear equations. The dVN equation has been introduced in [11, 20] as the dispersionless limit of the VN equation, which is the (2+1)-dimensional integrable generalization of the famous Korteweg–de Vries (KdV) equation [1–3, 5].

Let us consider a more specific situation in which the propagation of electromagnetic waves in the media discussed above exhibits also a slow variation along the y-axis, namely $\partial_y = \epsilon\partial_\eta$, where η is a slow variable defined by $y = \eta/\epsilon$ and ϵ is a small parameter. If one assumes that the functions S and u in the eikonal equation (9) have the following behaviour at small ϵ:

$$S = \frac{y}{2\epsilon} - \tilde{S}(x, \eta, \xi) \qquad n^2\left(x, \frac{\eta}{\epsilon}, \xi\right) = \frac{1}{4\epsilon^2} - q(x, \eta, \xi) \tag{18}$$

and the polynomial $\varphi\left(x, \frac{\eta}{\epsilon}, \xi; S_x, \epsilon S_\eta\right)$ has an appropriate behaviour as $\epsilon \to 0$, then in this limit, equations (9) and (10) are reduced to

$$\tilde{S}_\eta = \tilde{S}_x^2 + q \qquad \tilde{S}_\xi = \tilde{\varphi}(x, \eta, \xi; \tilde{S}_x) \tag{19}$$

where $\tilde{\varphi}$ is an odd order polynomial in $\tilde{S}_x$. Equations (19) describe propagation of the quasi-plane wave fronts $y = \text{const}$ in a medium with very large refractive index. Compatibility of equations (19) gives rise to the well-known dispersionless Kadomtsev–Petviashvili equation $q_\xi = \frac{3}{2}qq_x + \frac{3}{4}\partial_x^{-1}q_{\eta\eta}$ and the whole dKP hierarchy. The dKP equation is rather well studied (see, e.g., [11, 15] and references therein). The KP equation itself represents the most known (2+1)-dimensional integrable generalization of the KdV equation.

The dVN and the dKP equations being relevant in particular situations of propagation of waves in certain nonlinear media have an advantage of being solvable. Various methods have been applied to solve the dKP equation, including the quasiclassical $\bar{\partial}$-dressing method [18–20]. Here we will demonstrate that the dVN equation is amenable to this method too.

The quasiclassical $\bar{\partial}$-dressing method is based on the nonlinear Beltrami equation [18–20]

$$S_{\bar{\lambda}} = W(\lambda, \bar{\lambda}; S_\lambda) \tag{20}$$

where λ is the complex variable, $S_\lambda = \partial S/\partial\lambda$ and W is a certain function ($\bar{\partial}$-data). To construct integrable equations one has to specify the domain G (in the complex plane $\mathbb{C}$) of support for the function W ($W = 0, \lambda \in \mathbb{C}\backslash G$) and look for the solution of (20) in the form $S = S_0 + \tilde{S}$, where the function S_0 is analytic inside G, while $\tilde{S}$ is analytic outside G [18–20]. To obtain the dVN equation and the whole dVN hierarchy, we choose G as the ring $\mathcal{D} = \left\{\lambda \in \mathbb{C} : \frac{1}{a} < |\lambda| < a\right\}$, where a is an arbitrary real number ($a > 1$), and the function S_0 in the form

$$S_0 = \sum_{n=1}^{\infty} x_n\lambda^{2n-1} + \sum_{n=1}^{\infty} \bar{x}_n\lambda^{-2n+1}. \tag{21}$$

We also assume that

$$\tilde{S} = \sum_{n=0}^{\infty} \frac{S_{2n+1}^{(\infty)}}{\lambda^{2n+1}} \qquad \lambda \to \infty \tag{22}$$

'normalization' and

$$\tilde{S} = \sum_{n=0}^{\infty} \lambda^{2n+1} S_{2n+1}^{(0)} \qquad \lambda \to 0. \tag{23}$$

Note that a function $S(\lambda, \bar{\lambda}, x_n, \bar{x}_n)$ has the properties (21)–(23) if it obeys the constraints

$$S(-\lambda, -\bar{\lambda}) = -S(\lambda, \bar{\lambda}) \tag{24}$$

$$\bar{S}(\lambda, \bar{\lambda}) = S\left(\frac{1}{\bar{\lambda}}, \frac{1}{\lambda}\right). \tag{25}$$

The constraint (25) also implies that

$$S_{2n+1}^{(\infty)} = \bar{S}_{2n+1}^{(0)}.$$

An important property of the nonlinear $\bar{\partial}$-problem (20) is that the derivatives of S with respect to any independent variable x_n obeys the linear Beltrami equation

$$\left(\frac{\partial S}{\partial x_n}\right)_{\bar{\lambda}} = W'(\lambda, \bar{\lambda}; S_\lambda)\left(\frac{\partial S}{\partial x_n}\right)_\lambda \tag{26}$$

where $W'(\lambda, \bar{\lambda}; \phi) = \frac{\partial W}{\partial \phi}(\lambda, \bar{\lambda}; \phi)$. Equation (26) has two basic properties, namely, (1) any differentiable function of a solution is again a solution; (2) under certain mild conditions on W', a solution which is equal to zero at a certain point $\lambda_0 \in \mathbb{C}$, vanishes identically (Vekua's theorem) [25]. The use of these two properties allows us to construct algebraic equations of the form $\Omega(x_n, \bar{x}_n, S_{x_n}, S_{\bar{x}_n}) = 0$.

Denoting $x_1 = z$, choosing $x_2 = \bar{x}_2 = \xi$, taking into account that $S_z = \lambda + \tilde{S}_z$, $S_{\bar{z}} = \lambda^{-1} + \tilde{S}_{\bar{z}}$, $S_\xi = \left(\lambda^3 + \frac{1}{\lambda^3}\right) + \tilde{S}_\xi$, and using the properties of the linear Beltrami equation (26) mentioned above, one obtains

$$S_z S_{\bar{z}} = u \tag{27}$$

$$S_\xi = S_z^3 + S_{\bar{z}}^3 + V S_z + \bar{V} S_{\bar{z}} \tag{28}$$

where

$$u(z, \bar{z}, \xi) = 1 + S_{1\bar{z}}^{(\infty)} \qquad V = -3S_{1z}^{(\infty)} = -3\partial_{\bar{z}}^{-1} u_z. \tag{29}$$

Evaluating the terms of the order $1/\lambda$ on both sides of equation (28), one obtains the dVN equation

$$u_\xi = -3\left(u\partial_{\bar{z}}^{-1} u_z\right)_z - 3\left(u\partial_z^{-1} u_{\bar{z}}\right)_{\bar{z}}. \tag{30}$$

Considering higher 'times' $x_3, x_4, \ldots$ such that $x_n = \bar{x}_n$, one obtains the equations

$$S_{x_n} = \sum_{m=1}^{n} \left(u_m S_z^{2m-1} + \bar{u}_m S_{\bar{z}}^{2m-1}\right) \qquad n = 3, 4, \ldots. \tag{31}$$

Equations (31) together with (27) and (28) provide us with the infinite dVN hierarchy of equations.

Thus, the quasiclassical $\bar{\partial}$-dressing method allows us to treat the eikonal equation (27), equations (28) and (31) which describe its deformations and the corresponding dVN hierarchy for u, in a way similar to the dKP and d2DTL hierarchies [18–20]. These integrable dVN deformations of the eikonal equation (27) are quite special. In particular, they are characterized by the existence of an infinite set of integral quantities (integrals of motion for the dVN equation), which are preserved by these deformations. The simplest of them, for u such that $u \to 0$ as $|z| \to \infty$, is given by the total squared refractive index $C_1 = \iint u(z, \bar{z}, \xi)\, \mathrm{d}z \wedge \mathrm{d}\bar{z}$ ($C_{1\xi} = 0$).

Constraint (25) guarantees that the refractive index u is a real one. Indeed, taking the complex conjugation of equation (27), using the differential consequences (with respect to z and $\bar{z}$), of the above constraint and taking into account the independence of the lhs of equation (27) on λ,$\bar{\lambda}$, one obtains

$$\bar{u}(x_n) = \overline{S_z(\lambda, \bar{\lambda}, x_n)}\; \overline{S_{\bar{z}}(\lambda, \bar{\lambda}, x_n)} = S_{\bar{z}}\left(\frac{1}{\bar{\lambda}}, \frac{1}{\lambda}, x_n\right) S_z\left(\frac{1}{\bar{\lambda}}, \frac{1}{\lambda}, x_n\right) = u(x_n). \tag{32}$$

Moreover, this constraint implies that the function S is real valued on the unit circle $|\lambda| = 1$ ($\bar{S}(\lambda, \bar{\lambda}) = S(\lambda, \bar{\lambda})$, $|\lambda| = 1$).

Thus, the approach proposed provides us with the real-valued refractive index $u(x_n)$ and the phase function S together with their slow variations in the direction orthogonal to the plane x, y. Note it is not necessary that the refractive index $u = 4n^2$ should be positive. For certain media the product $\mu\epsilon_0$ can be negative as well [26].

The quasiclassical $\bar{\partial}$-dressing way provides also an effective way to construct the exact solutions of the dVN equations as in the dKP case [18–20]. The exact solutions of the dVN equation and their applications in geometrical optics will be discussed in a separate paper.

Finally we would like to note that the solutions of the nonlinear Beltrami equation (20) represent themselves the so-called quasiconformal mappings (for the dKP equation, see [18]). In the our case we have quasiconformal mappings of the ring $\mathcal{D}$, given by the function $S(\lambda, \bar{\lambda}, x_n)$, with the boundary conditions (22) and (23) and constraints (24) and (25). Thus the quasiclassical $\bar{\partial}$-dressing approach to the eikonal equation (27) and its deformations by dVN hierarchy establish a connection between the geometrical optics and the theory of the quasiconformal mappings on the plane [27, 28].

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